

## Dual Vectors

Believe it or not you have worked w/ dual vectors before (they are critical in defining a dot product). Chances are you didn't know because in  $\mathbb{R}^n$  they look just like vectors.

We define dual vectors by 3 conditions:

1. Dual vectors are straight directed objects defined at a point in space (hence they must live in a copy of  $\mathbb{R}^n$  or  $\mathbb{M}^n$  at each point which we call the cotangent space).
2. Dual vectors are invariant. But given a coordinate system, they can be expressed in terms of components and dual basis vectors which do transform.
3. Dual vectors linearly eat vectors and poop scalars:  $\omega(aV + bW) = a\omega(V) + b\omega(W) = \text{scalar}$



Wait, if vectors and dual vectors are invariant, then what the hell is a scalar?  
A scalar is an invariant whose explicit coordinate representation is also invariant.



Now for ordinary vectors we had the intuitive example  $dS = dx^{\hat{a}} \hat{e}_{(\hat{a})}$  to start with.  
For dual vectors we don't have such a starting point. So we need to be clever...

Consider condition 3 for dual vectors. We will define "eating" in terms of the basis vectors and dual basis vectors as follows:

dual basis vector

$$\hat{\theta}^{(n)} \hat{e}_{(v)} = \delta^n_v = \begin{cases} +1 & n=v \\ 0 & n \neq v \end{cases}$$

basis vector     a number!

Since  $\delta^n_v$  is a number (scalar) it must be invariant, Hence:

$$\hat{\theta}^{(n)} \hat{e}_{(v)} = \delta^n_v \rightarrow \hat{\theta}^{(n')} \hat{e}_{(v')} = \delta^{n'}_{v'}$$

We already know:  $\Lambda^v_{v'} \hat{e}_{(v)}$

So:  $\hat{\theta}^{(n')} \Lambda^v_{v'} \hat{e}_{(v)}$

$$\begin{aligned} \text{Guessing: } \Lambda^{n'}_m \hat{\theta}^{(n)} \Lambda^v_{v'} \hat{e}_{(v)} &= \Lambda^{n'}_m \Lambda^v_{v'} \underbrace{\hat{\theta}^{(n)} \hat{e}_{(v)}}_{\delta^n_v} \\ &= \underbrace{\Lambda^{n'}_m \Lambda^m_{v'}}_{\Lambda^{n'}_{v'}} = \delta^{n'}_{v'} \end{aligned}$$

So we now know:

$\begin{aligned} \hat{e}_{(n)} &\rightarrow \hat{e}_{(n')} = \Lambda^{n'}_m \hat{e}_{(n)} \\ \hat{\theta}^{(n)} &\rightarrow \hat{\theta}^{(n')} = \Lambda^{n'}_m \hat{\theta}^{(n)} \end{aligned}$	Transformation laws for basis vectors and dual basis vectors.
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But now that we know how dual basis vectors transform, we can use that an overall dual vector is invariant to determine how the components of a dual vector transform:

$\begin{aligned} V^m &\rightarrow V^{m'} = \Lambda^{m'}_m V^m \\ \omega_m &\rightarrow \omega_{m'} = \Lambda^m_{m'} \omega_m \end{aligned}$	Transformation law for components of vectors and dual vectors.
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We can now see the gory details of dual vectors eating vectors:

$$\begin{aligned}\omega(V) &= \omega_m \hat{\otimes}^{(M)} V^{\hat{\otimes} (N)} \\ &= \omega_m V^{\nu} \delta^{\mu}_{\nu} \\ &= \omega_m V^{\mu} = \underbrace{\omega_0 V^0 + \omega_1 V^1 + \omega_2 V^2 + \omega_3 V^3}_{\in \mathbb{R}}\end{aligned}$$

This might look like a dot product between two vectors, but this is actually a combination of a vector and a dual vector!

So what is the dot product?

Recall that the dot product combines two vectors to make a number:  $\bullet (V^{\mu}, V^{\nu}) \rightarrow \mathbb{R}$

In our language, what the dot product does is take one of the vectors and turn it into a dual vector, then let the dual vector eat the vector to make a number.

$$\bullet (V^{\mu}, V^{\nu}) = V_{\nu} V^{\mu} \in \mathbb{R}$$

But how do we take a given vector  $V^{\mu}$  and create a corresponding dual vector  $V_{\mu}$ ?

$$\begin{aligned}\text{We can take a hint from } \mathbb{R}^n: \bullet (V^i, V^j) &= V^i V^i + V^j V^j + \dots \\ &= \underbrace{\delta_{ij}}_{\text{metric on } \mathbb{R}^n} V^i V^j\end{aligned}$$

So for SR, we replace  $\delta_{ij} \rightarrow \eta_{\mu\nu}$

$$\text{Hence: } \bullet (V^{\mu}, V^{\nu}) = \eta_{\mu\nu} V^{\mu} V^{\nu} = V_{\nu} V^{\nu}$$

$$\begin{aligned}\text{In general: } V^{\mu} &\rightarrow V_{\mu} = \eta_{\mu\nu} V^{\nu} \\ V_{\mu} &\rightarrow V^{\mu} = \underbrace{\eta^{\mu\nu}}_{\eta^{-1}} V_{\nu}\end{aligned}$$

(note for SR  $\eta^{-1} = \eta$ , but this is not true in general!!)

So the metric (or its inverse) moves indices up or down (raise and lower) turning vectors into dual vectors and vice-versa.

Note: In all of this we are not changing coordinates!

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In the future we will largely drop all explicit reference to basis vectors  $\hat{e}_m$  and dual basis vectors  $\hat{\theta}^{(m)}$  since they take care of each other. We can instead just focus on the components.

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Tensors

1. Represent physical quantities that are invariant but when given an explicit coordinate representation will typically have components that transform.
2. Live in flat tangent or cotangent spaces (or tensor products of these) at each point.

In terms of a labelling scheme (tangent, cotangent) we have seen:

$(0,0)$     $(1,0)$     $(0,1)$     $(0,2)$     $(2,0)$   
 scalar   vectors   dual vectors   the metric   the inverse metric  
 $\eta_{\mu\nu}$     $\eta^{\mu\nu}$

In general you could have  $\begin{matrix} \text{upper} \\ \downarrow \\ (k, n) \\ \uparrow \\ \text{lower} \end{matrix}$  mixed indices, e.g.  $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta}$  (3,4) tensor

Though we will usually only deal with components, remember the full story is:

$$T = T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} \hat{e}_{\lambda} \otimes \hat{e}_{\mu} \otimes \hat{e}_{\nu} \otimes \hat{e}_{\alpha} \otimes \hat{\omega}^{\lambda} \otimes \hat{\omega}^{\beta} \otimes \hat{\omega}^{\gamma} \otimes \hat{\omega}^{\delta}$$

A popular definition: A tensor is something that transforms like a tensor.

A better definition: Tensors are multi-linear maps from the space of vectors and dual vectors into the reals, i.e.  $T(V, V, V, \omega, \omega, \omega, \omega) \rightarrow \mathbb{R}$

The latter definition is born out by a "well-fed" tensor:

Well-fed:  $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^{\lambda} V^{\beta} V^{\gamma} V^{\delta} \omega_{\mu} \omega_{\nu} \omega_{\alpha} \in \mathbb{R}$  no "free" indices

contrast with

Starving:  $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^{\lambda} V^{\beta} V^{\gamma} \omega_{\mu} \omega_{\nu} = F^{\alpha}{}_{\delta}$

Over-fed:  $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^{\lambda} V^{\beta} V^{\gamma} V^{\delta} \omega_{\mu} \omega_{\nu} \omega_{\alpha} = H^{\alpha}{}_{\delta}$